# PROBABILITY DISTRIBUTIONS ATTACHED TO GENERALIZED BERGMAN SPACES ON THE POINCARÉ DISK

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ABSTRACT. A family of probability distributions attached to a class of generalized weighted Bergman spaces on the Poincaré disk are introduced by constructing a kind of generalized coherent states. Their main statistical parameters are obtained explicitly. As application, photon number statistics related to coherent states under consideration are discussed.

## 1 INTRODUCTION

The *negative binomial states* are the field states that are superposition of the number states with appropriately chosen coefficients [1]. Precisely, these states are labeled by points $z$ of the complex unit disk $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$ , and are of the form

$$\mid z;2\beta >= (1-|z|^2)^{\beta} \sum_{k=0}^{+\infty} \left( \frac{\Gamma(2\beta+k)}{\Gamma(2\beta)k!} \right)^{\frac{1}{2}} z^k \mid k >, \tag{1.1}$$

where $2\beta > 1$ is a fixed parameter and $\mid k >$ are number states.

The probability of finding $k$ photons in the state (1.1) is given by the squared modulus of the projection of $\mid z\,;2\beta >$ onto the state $\mid k >$ as

$$|< k \mid z;2\beta >|^2 = (|z|^2)^k (1-|z|^2)^{2\beta} \frac{\Gamma(2\beta+k)}{\Gamma(2\beta)k!}, k = 0,1,2,\cdots \tag{1.2}$$

The latter is recognized as the *negative binomial distribution* $\mathcal{NB}\left(|z|^2, 2\beta\right)$ with $|z| < 1$ and $2\beta > 1$ as parameters [2]. Furthermore, the probability distribution (1.2) has a positive Mandel parameter and thereby the negative binomial states obey super-Poissonian statistics.

Note that the projection $< k \mid z\,;2\beta >$ in (1.2) can be rewritten as

$$< k \mid z,2\beta >= \left(K_\beta(z,z)\right)^{-\frac{1}{2}} h_k^\beta(z), \tag{1.3}$$

where

$$h_k^\beta(z) := \left( \frac{\Gamma(2\beta+k)}{\pi\Gamma(2\beta)k!} \right)^{\frac{1}{2}} z^k, k = 0,1,2,\cdots \tag{1.4}$$

and

$$K_\beta(z,w) := \pi^{-1}\left(1 - z\overline{w}\right)^{-2\beta}, z,w \in \mathbb{D} \tag{1.5}$$

are respectively an orthonormal basis and reproducing kernel of the weighted Bergman space

$$\mathcal{A}_{\beta,0}(\mathbb{D}) := \left\{ \varphi \in L^{2,\beta}(\mathbb{D}), \varphi \text{ holomorphic on } \mathbb{D} \right\}, \tag{1.6}$$

where $L^{2,\beta}(\mathbb{D})$ denotes the Hilbert space of functions $\varphi : \mathbb{D} \to \mathbb{C}$, which are square integrable with respect to the measure $(1-|z|^2)^{2\beta-2} d\mu$. Here $d\mu$ is the Lebesgue measure on $\mathbb{D}$.

By another hand, the Bergman space in (1.6) coincides with the null space

$$\mathcal{A}_\beta(\mathbb{D}) = \left\{ \varphi \in L^{2,\beta}(\mathbb{D}), \widetilde{H}_\beta\left[\varphi\right] = 0 \right\} \tag{1.7}$$

of the second order differential operator

$$\widetilde{H}_\beta := (1-|z|^2)^{-\beta} H_\beta (1-|z|^2)^{\beta}, \tag{1.8}$$

where the operator $H_\beta$ is given by

$$\frac{1}{4}H_\beta := -(1-|z|^2)^2\frac{\partial^2}{\partial z\partial\overline{z}} - \beta z\left(1-|z|^2\right)\frac{\partial}{\partial z} + \beta\overline{z}\left(1-|z|^2\right)\frac{\partial}{\partial\overline{z}} + \beta^2|z|^2 - \beta^2, \tag{1.9}$$

and constitutes (in suitable units and up to additive constant) a realization in $L^{2,0}(\mathbb{D})$ of the Schrödinger operator with uniform magnetic field in $\mathbb{D}$, with a field strength proportional to $\beta$ [3].

The spectrum of $\widetilde{H}_\beta$ in $L^{2,\beta}(\mathbb{D})$ consists of eigenvalues of infinite multiplicity *(hyperbolic Landau levels)* of the form:

$$\epsilon_m^\beta := 4(\beta-m)(1-\beta+m), m = 0,1,2,\cdots,\left[\beta-\frac{1}{2}\right] \tag{1.10}$$

provided that $2\beta > 1$. Here, $[\eta]$ denotes the largest integer not exceeding $\eta$. As for the Bergman space $\mathcal{A}_\beta(\mathbb{D})$ in (1.7) the eigenspace

$$\mathcal{A}_{\beta,m}(\mathbb{D}) := \left\{\varphi \in L^{2,\beta}(\mathbb{D}), \widetilde{H}_\beta[\varphi] = \epsilon_m^\beta\varphi\right\} \tag{1.11}$$

corresponding to the eigenvalue $\epsilon_m^\beta$ in (1.10) admits an orthogonal basis denoted $h_k^{\beta,m}(z), k = 0,1,2,\cdots$, given in terms of Jacobi polynomials as well as a reproducing kernel $K_{\beta,m}(z,w)$ in an explicit form. In this paper, we exploit these facts to construct a set of generalized coherent states as

$$| z; 2\beta, m >= \left(K_{\beta,m}(z,z)\right)^{-\frac{1}{2}}\sum_{k=0}^{+\infty}\frac{h_k^{\beta,m}(z)}{\sqrt{\rho_{\beta,m}(k)}} | k >, z \in \mathbb{D}, \tag{1.12}$$

where $\rho_{\beta,m}(k)$ denotes the norm square of the function $h_k^{\beta,m}(z)$ in $L^{2,\beta}(\mathbb{D})$. The states (1.12) enables us to attach to each eigenspace $\mathcal{A}_{\beta,m}(\mathbb{D})$ a kind of photon counting probability distribution in the same way as for the space $\mathcal{A}_\beta(\mathbb{D}) = \mathcal{A}_{\beta,0}(\mathbb{D})$ corresponding to the lowest hyperbolic Landau level $m = 0$. Indeed, for each fixed $m = 0,1,2,\cdots,\left[\beta-\frac{1}{2}\right]$ and $\lambda = |z|^2$, the probability mass function $P(X = k), k = 0,1,2,\cdots$ of this counting random variable $X$ is obtained as

$$p_k(\lambda, 2\beta; m) = \frac{\Gamma\left(1+\frac{1}{2}(m+k-|m-k|)\right)\Gamma\left(2\beta-m+\frac{1}{2}(|m-k|+k-m)\right)}{\Gamma\left(1+\frac{1}{2}(m+k+|m-k|)\right)\Gamma\left(2\beta-m-\frac{1}{2}(|m-k|+m-k)\right)}$$

$$\times (1-\lambda)^{2(\beta-m)}\lambda^{|m-k|}\left(P^{(|m-k|,2(\beta-m)-1)}_{\frac{1}{2}(m+k-|m-k|)}(1-2\lambda)\right)^2 \tag{1.13}$$

where $P_\eta^{(\tau,\varsigma)}(.)$ denotes the Jacobi polynomial [4]. The probability distribution (1.13) can be considered as a kind of generalized negative binomial probability distribution $X \sim \mathcal{NB}(\lambda, 2\beta, m)$ depending on an additional parameter $m = 0,1,2,\cdots,\left[\beta-\frac{1}{2}\right]$. Thus, we study the main properties of the family of probability distributions in (1.13) and we examine the quantum photon counting statistics with respect to the location inside the disk $\mathbb{D}$ of the point labeling the generalized coherent states $| z; 2\beta, m >$ in (1.12).

The paper is organized as follows. In Section 2, we recall briefly the negative binomial states as well as their principal statistical properties. Section 3 deals with some needed facts on the Shrödinger operator with magnetic field in the disk with an explicit description of some its needed eigenspaces. In Section 4, we associate to each generalized Bergman space a set of coherent states from which we obtain the announced probability distribution. In section 5 we give the main parameters of these probability distributions and we discuss the classicality/nonclassicality of the generalized coherent states with respect to the location of their labeling points inside the disk.

## 2 Negative binomial states

The *negative binomial states* are the field states that are superposition of the number states with appropriately chosen coefficients. They are intermediating states between a pure coherent state and a pure thermal state and reduce to Susskind-Glogower phases states for a particular limit of the parameter [4]. As mentioned above, these states are labeled by points $z$ with $|z| < 1$ and are of the form

$$\mid z; 2\beta >= (1-|z|^2)^{\beta} \sum_{k=0}^{+\infty} \left( \frac{\Gamma(2\beta+k)}{\Gamma(2\beta)k!} \right)^{\frac{1}{2}} z^k \mid k > \tag{2.1}$$

where $2\beta > 1$ is a fixed parameter and $\mid k >$ are number states.The states (2.1) are referred to as the negative binomial states since their photon probability distribution:

$$\Pr(X = k) = (|z|^2)^k \left(1-|z|^2\right)^{2\beta} \frac{\Gamma(2\beta+k)}{\Gamma(2\beta)k!} \tag{2.2}$$

obeys the negative binomial probability distribution, i.e., $X \sim \mathcal{NB}(\lambda, 2\beta)$ with parameters $\lambda = |z|^2$ and $2\beta > 1$. The mean number of photons and the variance are given by $E(X) = \lambda 2\beta(1-\lambda)^{-2}$ and $Var(X) = \lambda 2\beta(1-\lambda)^{-2}$. The Mandel $Q$ parameter for the negative binomial states equals $\lambda(1-\lambda)^{-1}$ and is always positive since $0 < \lambda < 1$. This means that photon statistics in the negative binomial states is always super-Poissonian.

According to [4], we should mention some limiting cases. For $\beta \to \infty$, $|z| \to 0$ but $\beta|z|^{-1} \to \mu$ the $\mathcal{NB}(\lambda, 2\beta)$ reduces to the Poisson distribution $\mathcal{P}(\mu)$ characteristic of the coherent states of the harmonic oscillator. For $\beta \to 0$, the photon number distribution $\mathcal{NB}(\lambda, 2\beta)$ reduces to the Bose-Einstein distribution. When $|z| \to 0$, $\mathcal{NB}(\lambda, 2\beta)$ goes to Dirac's measure $\delta_0$ and the negative binomial state in (2.1) goes to the vacuum state $\mid 0 >$ .

## 3 An orthonormal basis in $\mathcal{A}_{\beta,m}(\mathbb{D})$.

By [3] the Schrödinger operator on $\mathbb{D}$ with constant magnetic field of strength proportional to $\beta > 0$ can be written as :

$$\mathcal{L}_{\beta} := -(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \beta z \left(1-|z|^2\right) \frac{\partial}{\partial z} + \beta \bar{z} \left(1-|z|^2\right) \frac{\partial}{\partial \bar{z}} + \beta^2 |z|^2. \tag{3.1}$$

which is also called Maass Laplacian on the disk. A slight modification of $\mathcal{L}_B$ is given by the operator

$$H_{\beta} := 4\mathcal{L}_{\beta} - 4\beta^2 \tag{3.2}$$

acting in the Hilbert space

$$L^{2,0}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \to \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 \left(1-|z|^2\right)^{-2} d\mu(z) < +\infty \right\}, \tag{3.3}$$

The spectrum of $H_{\beta}$ in $L^{2,0}(\mathbb{D})$ consists of two parts: $(i)$ a continuous part $[1, +\infty[$ , $(ii)$ a finite number of eigenvalues of the form ([3]) :

$$\epsilon_m^{\beta} := 4(\beta - m)(1-\beta+m), m = 0, 1, 2, \cdots, \left[\beta - \frac{1}{2}\right] \tag{3.4}$$

with infinite degeneracy, provided that $2\beta > 1$. The eigenfunctions corresponding to eigenvalues in (3.4) are known as bound states. For our purpose, we shall consider the unitary equivalent realization $\widetilde{H}_{\beta}$ of the operator $H_{\beta}$ in the Hilbert space

$$L^{2,\beta}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \to \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 \left(1-|z|^2\right)^{2\beta-2} d\mu(z) < +\infty \right\}, \tag{3.5}$$

which is defined by

$$\widetilde{H}_{\beta} := \mathfrak{T}_{\beta}^{-1} H_{\beta} \mathfrak{T}_{\beta}, \tag{3.6}$$

where $\mathfrak{T}_{\beta} : L^{2,\beta}(\mathbb{D}) \to L^{2,0}(\mathbb{D})$ is the unitary transformation defined by the map $\varphi \mapsto (1-|z|^2)^{-\beta}\varphi$.

According to Eq. (3.6) the eigenspace of $H_{\beta}$ in $L^{2,0}(\mathbb{D})$, which corresponds to the eigenvalue $\epsilon_m^{\beta}$ in (3.4), is mapped isometrically via the transform $\mathfrak{T}_{\beta}$ onto the eigenspace

$$\mathcal{A}_{\beta,m}(\mathbb{D}) := \left\{ \Phi : \mathbb{D} \to \mathbb{C}, \Phi \in L^{2,\beta}(\mathbb{D}) \text{ and } \widetilde{H}_{\beta}\Phi = \epsilon_m^{\beta}\Phi \right\} \tag{3.7}$$

These eigenspaces will play a central role in this work and some of their spectral tools are summarized as follows:

**Proposition 3.1**. *Let $2\beta > 1$ and $m = 0, 1, 2, \cdots, \left[\beta - \frac{1}{2}\right]$. Then,*

$(i)$ *an orthogonal basis* of $\mathcal{A}_{\beta,m}(\mathbb{D})$ *is given by the set of functions*

$$\Phi_k^{\beta,m}(z) := |z|^{|m-k|}(1-|z|^2)^{-m} e^{-i(m-k)\arg z}$$
$$\times {}_2F_1\left(-m + \frac{m-k+|m-k|}{2}, 2\beta - m + \frac{|m-k|-m+k}{2}, 1+|m-k|; |z|^2\right) \tag{3.8}$$

$k = 0, 1, 2, \cdots$ *,where ${}_2F_1(a, b, c; x)$ is the Gauss hypergeometric function* [5].

$(ii)$ *the norm square $\rho_{\beta,m}(k)$ of the eigenfunction $\Phi_k^{\beta,m}$ in $L^{2,\beta}(\mathbb{D})$ is given by*

$$\rho_{\beta,m}(k) = \frac{\pi\left(\Gamma(1+|m-k|)\right)^2}{(2(\beta-m)-1)} \frac{\Gamma\left(m - \frac{|m-k|+m-k}{2} + 1\right)\Gamma\left(2\beta - m - \frac{|m-k|+m-k}{2}\right)}{\Gamma\left(m + \frac{|m-k|-m+k}{2} + 1\right)\Gamma\left(2\beta - m + \frac{|m-k|-m+k}{2}\right)}. \tag{3.9}$$

$(iii)$ *the diagonal of the reproducing kernel of the Hilbert $\mathcal{A}_{\beta,m}(\mathbb{D})$ is given by*

$$K_{\beta,m}(z,z) = \pi^{-1}(2\beta - 2m - 1)\left(1-|z|^2\right)^{-2\beta}, z \in \mathbb{D}. \tag{3.10}$$

**Proof.** For $(i)$, one can easily chek that the functions $\Phi_k^{\beta,m}(z)$ in (3.8) are of the form $\Phi_k^{\beta,m}(z) = \mathfrak{T}_B\left[\phi_k^{\beta,m}\right](z)$, where $\phi_k^{\beta,m}, k = 0, 1, 2, \cdots$, is an orthonormal basis of the space

$$\mathcal{A}_{\beta,m}^0(\mathbb{D}) := \left\{ \phi : \mathbb{D} \to \mathbb{C}, \phi \in L^{2,0}(\mathbb{D}) \text{ and } H_{\beta}\phi = \epsilon_m^{\beta}\phi \right\} \tag{3.11}$$

as discussed in [9, p. 9311], where the elements of the basis have been labeled by an integer $j \geq -m$ and therefore one has to take care of this by setting $k = j + m$. By the fact that $\mathfrak{T}_{\beta}$ is an isometry, on gets that $\Phi_k^{\beta,m}(z), k = 0, 1, 2, \cdots$, constitutes an orthonormal basis of $\mathcal{A}_{\beta,m}(\mathbb{D})$. For $(ii)$, the square norm of the eigenfunction $\phi_k^{\beta,m}$ in the Hilbert space $L^{2,0}(\mathbb{D})$ have been calculated in [6, p.9313] and remains the same for its image $\Phi_k^{\beta,m}$ in $L^{2,\beta}(\mathbb{D})$ under the unitary map $\mathfrak{T}_{\beta}$. For $(iii)$, it is not difficult to see that the reproducing kernel $K_{\beta,m}(z,w)$ of the Hilbert space $\mathcal{A}_{\beta,m}(\mathbb{D})$ reads

$$K_{\beta,m}(z,w) = (1-|z|^2)^{-\beta} K_{\beta,m}^0(z,w)(1-|w|^2)^{-\beta} \tag{3.12}$$

where $K_{\beta,m}^0(z,w)$ denotes the reproducing kernel of the Hilbert space $\mathcal{A}_{\beta,m}^0(\mathbb{D})$ in (3.11), whose diagonal term is given by the function [6, p.9313]:

$$K_{\beta,m}^0(z,z) = \pi^{-1}(2\beta - 2m - 1), z \in \mathbb{D}. \tag{3.13}$$

The proof of proposition is finished.□

We should note that in the case $m = 0$, the eigenspace $\mathcal{A}_{\beta,0}(\mathbb{D})$ coincides with the weighted Bergmann space on the disk defined in (1.6). Being motivated by this remark, the eigenspace

$\mathcal{A}_{\beta,m}(\mathbb{D})$ of $\widetilde{H}_\beta$ corresponding to the eigenvalue $\epsilon_m^\beta$ given in (3.2) will be called *generalized weighted Bergman space of index m*.

## 4 COHERENT STATES AND PROBABILITY DISTRIBUTIONS

In this section, we present a generalization of coherent states according to the procedure in [7]. For this, let $(X,\sigma)$ be a measure space and let $\mathcal{A}^2 \subset L^2(X,\sigma)$ be a closed subspace of infinite dimension. Let $\{f_n\}_{n=0}^{\infty}$ be an orthogonal basis of $\mathcal{A}^2$ satisfying, for arbitrary $u \in X$,

$$\omega(u) := \sum_{n=0}^{\infty} \rho_n^{-1} |f_n(u)|^2 < +\infty, \tag{4.1}$$

where $\rho_n := \|f_n\|^2_{L^2(X,\sigma)}$. Define

$$\mathfrak{K}(u,v) := \sum_{n=0}^{\infty} \frac{1}{\rho_n} f_n(u)\overline{f_n(v)},\ u,v \in X. \tag{4.2}$$

Then, $\mathfrak{K}(u,v)$ is a reproducing kernel, $\mathcal{A}^2$ is the corresponding reproducing kernel Hilbert space and $\omega(u) := \mathfrak{K}(u,u)$, $u \in X$.

**Definition. 4.1**. *Let $\mathcal{H}$ be a Hilbert space with* $\dim \mathcal{H} = \infty$ *and $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. The coherent states labeled by points $u \in X$ are defined as the ket-vectors $\vartheta_u \equiv\mid u> \in \mathcal{H}$ :*

$$\vartheta_u \equiv\mid u >:= (\omega(u))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{f_n(u)}{\sqrt{\rho_n}} \phi_n. \tag{4.3}$$

Now, by Definition 4.1, it is straightforward to show that $< u \mid u >= 1$ and the coherent state transform $W : \mathcal{H} \to \mathcal{A}^2 \subset L^2(X,\sigma)$ defined by

$$W[\phi](u) := (\omega(u))^{\frac{1}{2}} < \vartheta_u \mid \phi > \tag{4.4}$$

is an isometry. Thus, for $\phi,\psi \in \mathcal{H}$, we have

$$< \phi \mid \psi >_{\mathcal{H}} = < W[\phi] \mid W[\psi] >_{L^2(X,\sigma)} = \int_X d\sigma(u)\,\omega(u) < \phi \mid \vartheta_u >< \vartheta_u \mid \psi > .$$

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac's bra-ket notation as

$$\mathbf{1}_{\mathcal{H}} = \int_X d\sigma(u)\,\omega(u) \mid u >< u \mid, \tag{4.5}$$

and where $\omega(u)$ appears as a weight function.

Now, we are in position to construct for each hyperbolic Landau level $\epsilon_m^B$ given in (3.4) a set of generalized coherent states according to formula (4.3) as

$$\mid z, 2\beta, m >:= \left(K_{\beta,m}(z,z)\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{\Phi_k^{\beta,m}(z)}{\sqrt{\rho_{\beta,m}(k)}} \mid k, \alpha > \tag{4.6}$$

with the following meaning:

· $(X,\sigma) = (\mathbb{D}, \left(1-|z|^2\right)^{2\beta-2} d\mu(z))$, $d\sigma(z) = (1-|z|^2)^{2\beta-2} d\mu(z)$, $d\mu(z)$ being the Lebesgue measure on $\mathbb{D}$,

· $\mathcal{A}^2 := \mathcal{A}_{\beta,m}(\mathbb{D})$ denotes the eigenspace of $\widetilde{H}_\beta$in $L^{2,\beta}(\mathbb{D})$,

· $K_{\beta,m}(z,z) = \pi^{-1}(2\beta - 2m - 1)(1-|z|^2)^{-2\beta}$,

· $\Phi_k^{\beta,m}(z)$ are the eigenfunctions given by (3.8) in terms of the Gauss hypergeometric function ${}_2F_1(.)$

· $\rho_{\beta,m}(k)$ being the norm square of $\Phi_k^{\beta,m}$ given in (3.9),

$\cdot$ $\mathcal{H}$:=$L^2(\mathbb{R}_+^*, x^{-1}dx)$ is the Hilbert space carrying the coherent states (5.6),

$\cdot \mid k, \alpha >\equiv \psi_k^\alpha, k = 0, 1, 2, \cdots$ ,is the complete orthonormal basis of $L^2(\mathbb{R}_+^*, x^{-1}dx)$ consisting of functions given by [8]:

$$\psi_k^\alpha(x) := \left(\frac{\Gamma(k+2\alpha)}{k!}\right)^{-\frac{1}{2}} x^\alpha \exp\left(-\frac{1}{2}x\right) L_k^{(2\alpha-1)}(x), x \in \mathbb{R}_+^*$$

where $L_k^{(\eta)}(.)$ denotes the generalized Laguerre polynomial [5].

**Definition 4.2**. *For each fixed* $m = 0, 1, 2, \cdots, \left[\beta - \frac{1}{2}\right]$ *. The coherent states* $(\mid z, \beta, m >)_{z\in\mathbb{D}}$ *associated with the generalized Bergman space* $\mathcal{A}_{B,m}(\mathbb{D})$ *are defined as a superposition of the basis* $\psi_k^\alpha$ of the Hilbert space $L^2(\mathbb{R}_+^*, x^{-1}dx)$ *through the wave functions*

$$< x \mid z, 2\beta, m >:= \frac{\sqrt{\pi}}{\sqrt{2\beta - 2m - 1}}\left(1 - |z|^2\right)^{\beta - m}$$

$$\times \sum_{k=0}^{+\infty} \frac{|z|^{|m-k|} e^{-i(m-k)\arg z}}{\sqrt{\rho_{\beta,m}(k)}} P_{\min(m,k)}^{(|m-k|, 2(\beta-m)-1)}\left(1 - 2|z|^2\right) \psi_k^\alpha(x), x \in \mathbb{R}_+^*. \tag{4.7}$$

Now, in view of (4.7) the projection of the coherent states $\mid z, \beta, m >$ onto the state $\psi_k^\alpha$ is given by the scalar product

$$\langle z, 2\beta, m \mid \psi_k^\alpha\rangle_{\mathcal{H}} = \left(K_{\beta,m}(z,z)\right)^{-\frac{1}{2}} \frac{\Phi_k^{\beta,m}(z)}{\sqrt{\rho_k^{\beta,m}}}, k = 0, 1, 2, \cdots . \tag{4.8}$$

Therefore, the squared modulus of $\langle z, 2\beta, m \mid \psi_k^\alpha\rangle_{\mathcal{H}}$ gives the probability that $k$ photons will be found in the coherent state $\mid z, 2\beta, m >$ .This leads to the mass distribution

$$p_k\left(|z|^2, 2\beta, m\right) := \left|\langle z, 2\beta, m \mid \psi_k^\alpha\rangle_{\mathcal{H}}\right|^2, \quad k = 0, 1, 2, \ldots, \tag{4.9}$$

which is denoted $p_k(\lambda, 2\beta, m)$ with $\lambda = |z|^2$. Being motivated by this quantum probability, we then write:

**Definition 4.3**. *For each fixed* $m = 0, 1, 2, \cdots, \left[\beta - \frac{1}{2}\right]$ *the discrete random variable* $X$ *with the probability distribution*

$$p_k(\lambda, 2\beta; m) := \gamma_{\beta,m,k}(1-\lambda)^{2(\beta-m)} \lambda^{|m-k|}\left(P_{\frac{1}{2}(m+k-|m-k|)}^{(|m-k|, 2(\beta-m)-1)}(1-2\lambda)\right)^2$$

*with*

$$\gamma_{\beta,m,k} := \frac{\Gamma\left(1 + \frac{1}{2}(m+k-|m-k|)\right)\Gamma\left(2\beta - m + \frac{1}{2}(|m-k| + k - m)\right)}{\Gamma\left(1 + \frac{1}{2}(m+k+|m-k|)\right)\Gamma\left(2\beta - m - \frac{1}{2}(|m-k| + m - k)\right)} \tag{4.10}$$

*and denoted by* $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ *,* $\lambda > 0$ and $2\beta > 1$ *will be called the extended negative binomial probability distribution attached to the generalized Bergman space* $\mathcal{A}_{\beta,m}(\mathbb{D})$.

**Remark 4.1.** Note that for $m = 0$, the above expression in (5.10) reduces to

$$p_k(\lambda, 2\beta; 0) = (1-\lambda)^{2\beta}\lambda^k \frac{\Gamma(2\beta + k)}{k!\Gamma(2\beta)}, k = 0, 1, 2, \cdots \tag{4.11}$$

which is the standard negative binomial distribution $\mathcal{NB}(\lambda, 2B)$ with parameter $\lambda$ and $2\beta$ in (2.1)

**Remark 4.2.** We should note that expression of the mass distribution $p_k(\lambda, 2\beta, m)$ in (4.10) may also appear when calculating Franck-Condon factors in special case of molecular vibration described by the Morse potential [9, p.6].

## 5 The generating function of $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ and photon number statistics

The purpose of this section is to give some essential parameters of $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ . We first determine the generating function

$$G_X^m(\xi) = \sum_{k=0}^{+\infty} \xi^k p_k(\lambda, 2\beta; m) \tag{5.1}$$

as a convenient way to obtain information about this random variable.

**Proposition 6.1.** *Let* $m = 0, 1, 2, \cdots, \left[\beta - \frac{1}{2}\right]$ . *Then the generating function of the random variable* $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ *is given by*

$$G_X^m(\xi) = \left(\frac{1-\lambda}{1-\lambda\xi}\right)^{2\beta} \left(\frac{(\lambda-\xi)(1-\lambda\xi)}{(1-\lambda)^2}\right)^m P_m^{(2(\beta-m)-1,0)}\left(1 + \frac{2\xi(1-\lambda)^2}{(\lambda-\xi)(1-\lambda\xi)}\right) \tag{5.2}$$

**Proof.** The integer $m = 0, 1, \cdots, \left[\beta - \frac{1}{2}\right]$ being fixed, we start by writing the generating function of $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ according to (5.1) and we make use of definition (4.3), we have that

$$G_X^m(\xi) = \sum_{k=0}^{+\infty} \gamma_{\beta,m,k} \xi^k (1-\lambda)^{2(\beta-m)} \lambda^{|m-k|} \left(P_{\frac{1}{2}(m+k-|m-k|)}^{(|m-k|, 2(\beta-m)-1)}(1-2\lambda)\right)^2 \tag{5.3}$$

We split this sum into two part as

$$G_X^m(\xi) = \mathcal{G}_{\beta,m,\lambda}^{(<\infty)}(\xi) + \mathcal{G}_{\beta,m,\lambda}^{(\infty)}(\xi) \tag{5.4}$$

where $\mathcal{G}_{\beta,m,\lambda}^{(<\infty)}(\xi)$ denotes a finite sum given by

$$\begin{aligned}\mathcal{G}_{\beta,m,\lambda}^{(<\infty)}(\xi) \quad : \quad &= \sum_{j=0}^{m-1} (1-\lambda)^{2(\beta-m)} \xi^k \Big(\frac{k!}{m!} \frac{\Gamma(2\beta-m)}{\Gamma(2\beta-2m+k)} \lambda^{m-k} \left(\left(P_k^{(m-k, 2(\beta-m)-1)}(1-2\lambda)\right)^2\right) \\ &- \frac{m!}{k!} \frac{\Gamma(2\beta-2m+k)}{\Gamma(2\beta-m)} \lambda^{k-m} \left(\left(P_m^{(k-m, 2(\beta-m)-1)}(1-2\lambda)\right)^2\right)\end{aligned}$$

and $\mathcal{G}_{\beta,m,\lambda}^{(\infty)}(\xi)$ denotes the following infinite sum:

$$\mathcal{G}_{\beta,m,\lambda}^{(\infty)}(\xi) := \sum_{k=0}^{+\infty} \xi^k \frac{m!}{k!} \frac{\Gamma(2\beta-2m+k)}{\Gamma(2\beta-m)} (1-\lambda)^{2(\beta-m)} \lambda^{k-m} \left(\left(P_m^{(k-m, 2(\beta-m)-1)}(1-2\lambda)\right)^2\right) \tag{5.5}$$

Noting that the finite sum $\mathcal{G}_{\beta,m,\lambda}^{(<\infty)}(\xi)$ contains the following difference

$$\frac{\lambda^{m-k}\left(P_k^{(m-k, 2(\beta-m)-1)}(1-2\lambda)\right)^2}{(k!\Gamma(2\beta-m))^{-1} m!\Gamma(2\beta-2m+k)} - \frac{\lambda^{k-m}\left(P_m^{(k-m, 2(\beta-m)-1)}(1-2\lambda)\right)^2}{(m!\Gamma(2\beta-2m+k))^{-1} k!\Gamma(2\beta-m)}. \tag{5.6}$$

The latter suggests to make use of the identity $([10], p.63)$:

$$\frac{\Gamma(n+1)}{\Gamma(n-l+1)\, l!} P_n^{(-l,\nu)}(u) = \frac{\Gamma(n+\nu+1)}{l!\Gamma(n+\nu-l+1)} \left(\frac{u-1}{2}\right)^l P_{n-l}^{(l,\nu)}(u),\ 1 \le l \le n \tag{5.7}$$

for $k = n$ , $l = k - m$, $u = 1 - 2\lambda$ and $\nu = 2(\beta - m) - 1$. We then write

$$P_k^{(m-k,\nu)}(1-2\lambda) = \frac{m!\Gamma(2\beta-2m+k)}{(-1)^{m-k} k!\Gamma(2\beta-m)\lambda^{m-k}} P_m^{(k-m,\nu)}(1-2\lambda). \tag{5.8}$$

After calculation, we obtain that $\mathcal{G}^{(<\infty)}_{\beta,m,\lambda}(\xi)=0$. Therefore, it remains to calculate the infinite sum which reads

$$\mathcal{G}^{(<\infty)}_{\beta,m,\lambda}(\xi)=\Upsilon^{1}_{\beta,m}(\lambda,\xi)\sum_{k=0}^{+\infty}\frac{\Gamma(2\beta-2m+k)}{k!}(\xi\lambda)^{k-m}\left(P_m^{(k-m,\nu)}(1-2\lambda)\right)^2 \tag{5.9}$$

where the prefactor is given by

$$\Upsilon^{1}_{\beta,m}(\lambda,\xi):=\frac{m!\,(1-\lambda)^{2(\beta-m)}\,\xi^m}{\Gamma(2\beta-m)} \tag{5.10}$$

If we put $\tau=\xi\lambda$ and $k-m=s$, we will need to calculate the sum

$$\mathcal{S}:=\sum_{s\geq -m}\frac{\Gamma(2\beta-m+s)}{(s+m)!}\tau^s\left(P_m^{(s,\nu)}(u)\right)^2 \tag{5.11}$$

where $u=1-2\lambda$ and $\nu=2(\beta-m)-1$. Once again, we make use of the identity (5.7) to rewrite the sum (5.11) as follows

$$\mathcal{S}=\Upsilon^{2}_{\beta,m}(u,\tau)\sum_{j=0}^{+\infty}\frac{j!}{(\nu+1)_j}\left(P_j^{(m-j,\nu)}(u)\right)^2\left(\frac{4\tau}{(u-1)^2}\right)^j \tag{5.12}$$

where the prefactor

$$\Upsilon^{2}_{\beta,m}(u,\tau):=\left(\frac{\Gamma(2\beta-m)}{m!}\right)^2\tau^{-m}\left(\frac{u-1}{2}\right)^{2m} \tag{5.13}$$

Making use of the following identity due to Srivastava and Rao [11, p. 1329]:

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{n!t^n}{(1+\beta_0)_n}P_m^{(\gamma-n,\beta_0)}(x)\,P_m^{(\gamma-n,\beta_0)}(y)&=\frac{(1-t)^{\gamma}}{\left(1-\frac{(x-1)(y-1)t}{4}\right)^{1+\gamma+\beta_0}}\\ \times{}_2F_1\left(1+\gamma+\beta_0,-\gamma,1+\beta_0;\frac{-(x+1)(y+1)t}{(1-t)(4-(x-1)(y-1)t)}\right)&\end{aligned} \tag{5.14}$$

for $t=\frac{4\tau}{(u-1)^2}$, $x=y=u$ , $\beta_0=\nu,\gamma=m$ and $n=j$, we obtain after computation and summarizing up the above steps

$$\begin{aligned}G_X^m(\xi)\quad=\quad&\frac{\Gamma(2\beta-m)}{m!\Gamma(2(\beta-m))}\left(\frac{1-\lambda}{1-\lambda\xi}\right)^{2\beta}\left(\frac{(\lambda-\xi)(1-\lambda\xi)}{(1-\lambda)^2}\right)^m\\ &\times{}_2F_1\left(-m,2\beta-m,2(\beta-m);\frac{-\xi(1-\lambda)^2}{(\lambda-\xi)(1-\lambda\xi)}\right)\end{aligned} \tag{5.15}$$

Finally, by the help of the relation [5]:

$${}_2F_1\left(k+\nu+\varrho+1,-k,1+\nu;\frac{1-t}{2}\right)=\frac{k!\Gamma(1+\nu)}{\Gamma(k+1+\nu)}P_k^{(\nu,\varrho)}(t)$$

connecting the hypergeometric function ${}_2F_1(.)$ with the Jacobi polynomial $P_k^{(\nu,\varrho)}(.)$, we arrive at the announced expression of the generating function $G_X^m(\xi)$. This ends the proof of Proposition 5.1. □

**Remark 5.1.** Note that for $m=0$, the expression in (6.2) reduces to

$$G_X^0(\xi)=\left(\frac{1-\lambda\xi}{1-\lambda}\right)^{-2\beta} \tag{5.16}$$

which is the well known characteristic function of standard negative binomial distribution $\mathcal{NB}(\lambda,2\beta)$with parameter $\lambda$ and $2\beta$.

**Corollary 5.1.** *Let* $m = 0, 1, 2, \cdots \left[\beta - \frac{1}{2}\right]$ *Then the mean value and the variance of the random variable* $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ *are respectively given by*

$$\begin{aligned} E(X) &= \frac{2\lambda\beta}{1-\lambda} + m && (5.17) \\ Var(X) &= \frac{2\lambda}{(1-\lambda)^2} + \frac{m}{(1-\lambda)^2}\lambda\left(\beta - 2 - \frac{\lambda}{2}\right) \end{aligned}$$

**Proof.** We make use of the expression of the generating function $G_X^m(\xi)$ obtained in Proposition 6.1 to derive the mean value the mean value of the random variable $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ through the relation

$$E(X) = \frac{\partial}{\partial \xi}\left(G_X^m(\xi)\right)|_{\xi=1} \tag{5.18}$$

Straightforward calculations give

$$\frac{\partial}{\partial \xi}\left(G_X^m(\xi)\right)|_{\xi=1} = \frac{2\lambda\beta}{1-\lambda} + m. \tag{5.19}$$

The variance is also obtained by using the well known fact that

$$Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 \tag{5.20}$$

where $E\left(X^2\right)$ can also be obtained from the the generating function $G_X^m(\xi)$ as

$$E\left(X^2\right) = \frac{\partial}{\partial \xi}\left(G_X^m(\xi)\right)|_{\xi=1} + \frac{\partial^2}{\partial \xi^2}\left(G_X^m(\xi)\right)|_{\xi=1} \tag{5.21}$$

After tedious calculations we obtain that

$$\begin{aligned} E\left(X^2\right) = \frac{2\beta(2\beta+1)\lambda^2 + 4m(1-\lambda)\lambda\beta}{(1-\lambda)^2} + m\left(m - 1 - \frac{2\lambda}{(1-\lambda)^2}\right) && (5.22) \\ + \frac{m\lambda(2\beta-\lambda)}{2(1-\lambda)^2}. \end{aligned}$$

Substituting (5.22) and (5.19) in (5.20), we arrive at

$$Var(X) = \frac{2\lambda\beta}{(1-\lambda)^2} + \frac{m}{(1-\lambda)^2}\lambda\left(\beta - 2 - \frac{\lambda}{2}\right)$$

This ends the proof of the corollary.□

**Remark 5.2.** For $m = 0$, the result of corollary 6.1 reads $E(X) = 2\beta\lambda(1-\lambda)^{-2}$ and $Var(X) = 2\beta\lambda(1-\lambda)^{-2}$ which are known parameters of the standard negative binomial probability distribution.

## 6 Photon counting statistics

To define a measure of non classicality of a quantum states one can follow several different approach. An earlier attempt to shed some light on the non-classicality of a quantum state was pioneered by Mandel [12], who investigated radiation fields and introduced the parameter

$$Q = \frac{Var(X)}{E(X)} - 1, \tag{6.1}$$

to measure the deviation of the photon number statistics from the Poisson distribution , characteristic of coherent states. Indeed, $Q = 0$ characterize Poissonian statistics. If $Q < 0$ we have *sub-Poissonian* statistics otherwise, statistics are *super-Poissonian.*

In our context, as mentioned in section 1, if $m = 0$ then $X \sim \mathcal{NB}(\lambda, 2\beta)$ obeys the negative binomial distribution and the corresponding photon counting statistics are super-Poissonian. For $m \neq 0$ we make use of the statistical parameters obtained in corollary 5.1 to calculate

Mandel parameter $Q(X)$ corresponding the random variable $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ and we summarize the discussion with respect to the sign of $Q(X)$ in the following statement:

**Proposition 6.1**. *Let $m = 1, 2, \cdots, \left[\beta - \frac{1}{2}\right]$. Then, the photon counting statistics are :*

*$(i)$ sub-Poissonian for points $z \in \mathfrak{D}$ belonging to the open disk $\mathfrak{D}\left(0, r_{\beta,m}\right)$ or radius*

$$r_{\beta,m} := \frac{-m\beta + \sqrt{(\beta^2 - 6)\, m^2 + 8\beta m}}{4\beta - 3m} \tag{6.2}$$

*For such labeling point $z$ the states $\mid z, \beta, m >$ are non-classical.*

*$(ii)$ Poissonian for points $z$ of the the boundary disk $\partial\mathfrak{D}\left(0, r_{\beta,m}\right)$. Here the states $\mid z, 2\beta, m >$ becomes pure coherent states.*

*$(iii)$ Super-Poissonian for $z \in \mathbb{D} \setminus \mathfrak{D}\left(0, r_{\beta,m}\right)$. For such points the states $\mid z, 2\beta, m >$ may describe thermal (or chaotic) light.*

**Proof.** Making use of corollary 6.1, the Mandel parameter (6.1) corresponding to the random variable $X \sim \mathcal{NB}(\lambda, 2\beta; m)$ has the following expression

$$Q(X) = \frac{(4\beta - 3m)\, \lambda^2 + 2\beta\lambda m - 2m}{2(1-\lambda)\,(2\beta\lambda - m\lambda + m)}. \tag{6.3}$$

We look at the roots of the equation

$$(4\beta - 3m)\, \lambda^2 + 2\beta\lambda m - 2m = 0 \tag{6.4}$$

with respect to the variable $\lambda$ with $0 < \lambda < 1$. The discriminant $\Delta' = \beta^2 m^2 + 2m\,(4\beta - 3m) > 0$ since $0 \leq m \leq \left[\beta - \frac{1}{2}\right]$ and one can easily see that roots of Eq. (6.4) are of the form:

$$\lambda_{\pm}(\beta, m) := \frac{-mB \pm \sqrt{(\beta^2 - 6)\, m^2 + 8\beta m}}{4\beta - 3m} \tag{6.5}$$

But the only *admissible* root in the sense that it belongs to the interval $]0, 1[$ is $\lambda_+$. We put $r_{\beta,m} := \lambda_+(\beta, m)$ and the assertions $(i), (ii)$ and $(iii)$ follow by discussing the sign of the parameter $Q(X)$ in (6.3). □

## References


[1] V. V. Dodonov: Nonclassical' states in quantum optics: a 'squeezed review of the first 75 years, *J. opt. B: Quantum Semiclass. opt*. **4** (2002), 1-33.

[2] W. Feller, An introduction to probability: theory and its applications, Vol1 2nd ed., John Wiley, 1957

[3] E. V. Ferapontov and A.P. Veselov: Integrable Schrodinger operators with magnetic fields: Factorization method on curved surfaces, *J. Math. Phys.* **42** (2001), 590-607.

[4] H-C Fu and R. Sasaki, Negative binomial states of quantized Radiation fields, Preprint YIPT-96-54, arXiv: quant-ph/ 9610024v1.

[5] Gradshteyn I S and Ryzhik I M, "Table of Integrals, Series and Products", Academic Press, INC (1980)

[6] Mouayn Z, Coherent states attached to Landau levels on the Poincaré disk, *J. Phys. A : Math. Gen*. vol. 38, no 42, pp. 9309-9316 (2005)

[7] K. Thirulogasantar and N. Saad: Coherent states associated with the wavefunctions and the spectrum of the isotonic oscillator, *J. Phys. A: Math. Gen.* **37** (2004), 4567-4577.

[8] Titchmarsh E C, Introduction to the theory of Fourier integrals, 2 End ed., Clarendon Press (1950)

[9] J. Koch, M. Semmelhack, F. Von Oppen and A. Nitzan, Current induced nonequilibrium vibrations in single-molecules devices, Physical Rev. B **73**, 155306 (2006)

[10] A. Szegö, Orthogonal polynomials. American Mathematical Society; Providence, R.I. (1975)

[11] A. Srivastava and A. B. Rao, A polynomial of the form $F_4$, *Indian Jour. Pure and App. Math*, 6(1), pp.326-1339 (1975)

[12] L. Mandel, Sub-poissonian photon statistics in resonance fluorescence. *Opt. Lett.*, 4:205–207, 1979.

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